The wild McKay correspondence for $\mathbb{Z}/p^n\mathbb{Z}$ and its application

Mahito Tanno

Department of Mathematics, Osaka University
(joint work with Takehiko Yasuda)

Introduction/Motivation

Main interest in our study is the $\boldsymbol{v}$-functions, which have an important role in the wild McKay correspondence; and it is considered as a common generalization of the *age* invariant in the tame McKay correspondence and of the *Artin conductor*, see Wood–Yasuda [4] for detail. The wild McKay correspondence theorem claims that the *stringy motive* of associated quotient varieties can be computed via the $\boldsymbol{v}$-function. Furthermore, we can estimate its discrepancy, see Yasuda [5, Proposition 2.1].

We give an explicit formula of the $\boldsymbol{v}$-functions for the case of cyclic groups of prime power order; and then apply it to study singularities of quotient varieties by cyclic groups of prime square order. All results in this poster can be found at the preprint [3], joint work with Takehiko Yasuda.

We denote by k an algebraically closed field of characteristic $p > 0$ and put $K = k((t))$. We also set $G = \mathbb{Z}/p^n\mathbb{Z}$. We identify a G-representation V of dimension d with $\mathbb{A}^d_k$.

The wild McKay correspondence

The McKay correspondence theorem relates an invariant of a representation V of a finite group G with an invariant of the associated quotient variety $X = V/G$. We use motivic invariants, which originates the work of Batyrev [1] and Denef–Loeser [2] in characteristic zero. Yasuda [6] generalized their results to arbitrary characteristics:

Theorem 1 (Yasuda [6, Corollary 16.3]) *Let k be a perfect field and G a finite group. Assume that G acts on $\mathbb{A}^d_k$ linearly and effectively and that G has no pseudo-reflection. Put $X \coloneqq \mathbb{A}^d_k/G$ Then we have*

$$M_{\mathrm{st}}(X) = \int_{G\text{-Cov}(D)} \mathbb{L}^{d-\boldsymbol{v}}. \tag{1}$$

Here $M_{\mathrm{st}}(X)$ denotes the stringy motive *of the quotient variety X, G-Cov(D) denotes the moduli space of G-covers of $D \coloneqq \operatorname{Spec} k[[t]]$ and $\boldsymbol{v}\colon\ G\text{-Cov}(D) \to \mathbb{Q}$ is the $\boldsymbol{v}$-function associated to the given G-action.*

G-covers

A G-cover of $D = \operatorname{Spec} k[[t]]$ is a normalization of D in an étale G-cover of $D^* = \operatorname{Spec} K$. Étale G-covers of D^* are parameterized by p^m-cyclic extensions of K $(0 \le m \le n)$. The Artin–Schreier–Witt theory controls such extensions; we have $G\text{-Cov}(D^*) = W_n(K)/\wp(W_n(K))$, where $W_n(K)$ denotes the ring of Witt vectors of length n over K and $\wp$ the Artin–Schreier morphism. Moreover, we can find "good" representatives of elements of $W_n(K)/\wp(W_n(K))$, which are called *reduced*. The components of each reduced Witt vector $\boldsymbol{f} = (f_l)_l$ are Laurent polynomials which has no positive degree term such that $p \nmid \operatorname{ord} f_l$ for all l.

We denote by $G\text{-Cov}(D; \boldsymbol{j})$ the set of G-covers whose corresponding reduced Witt vector $(f_l)_l$ satisfy $-\boldsymbol{j} = (-j_l)_l = (\operatorname{ord} f_l)_l$. We can induce a geometric structure on each $G\text{-Cov}(D^*; \boldsymbol{j})$ and hence we get a stratification

$$G\text{-Cov}(D^*; \boldsymbol{j}) = \prod_{j_l \ne -\infty} \mathbb{G}_{m,k} \times \mathbb{A}_k^{j_l - 1 - \lfloor j_l/p \rfloor}, \quad G\text{-Cov}(D^*) = \bigsqcup_{\boldsymbol{j}} G\text{-Cov}(D^*; \boldsymbol{j}).$$

$\boldsymbol{v}$-functions

Let E be a G-cover of D induced from $E^* \in G\text{-Cov}(D; \boldsymbol{j})$ and $\mathcal{O}_E$ the coordinate ring of E. Then the direct sum $\mathcal{O}_E^{\oplus d}$ has two G-action. One is the diagonal action induced from G-action on $\mathcal{O}_E$. The other is given by the composition $G \to GL(d, k) \hookrightarrow GL(d, \mathcal{O}_E)$, where left map is associated to the given G-action on $\mathbb{A}^d_k$. We define the *tuning module* $\Xi_E \subset \mathcal{O}_E^{\oplus d}$ to be the submodule of elements on which the two actions coincide. Then we define $\boldsymbol{v}(E) \coloneqq \operatorname{colength}(\mathcal{O}_E \cdot \Xi_E \subset \mathcal{O}_E^{\oplus d})/\#G$.

Assume that $E^* = \operatorname{Spec} L$ is connected and V is indecomposable. This case is essential; the case of non-connected covers is reduced to the case of smaller group; the case of decomposable representation is reduced to the case of indecomposable because of additivity of $\boldsymbol{v}$-functions. In this situation, we find out that the $\boldsymbol{v}$-function can be computed in terms of the ramification jumps of the corresponding field extension L/K:

Theorem 2 (T.–Yasuda [3, Theorem 3.11]) *Let $E^* = \operatorname{Spec} L$ be a G-cover of $D^* = \operatorname{Spec} K$. Assume that the G-extension L/K is defined by an equation $\wp(\boldsymbol{g}) = \boldsymbol{f}$, where $\boldsymbol{f} = (f_l)_l$ is a reduced Witt vector of* $\operatorname{ord} f_l = -j_l$. *Put*

$$u_i = \max\{p^{n-1-m} j_m \mid m = 0, 1, \dots, i-1\},$$
$$l_i = u_0 + (u_1 - u_0)p + \cdots + (u_i - u_{i-1})p^i.$$

Then

$$\boldsymbol{v}(E) = \sum_{\substack{0 \le i_0 + p i_1 + \cdots + p^{n-1} i_{n-1} < d, \\ 0 \le i_0, i_1, \dots, i_{n-1} < p}} \left\lceil \frac{i_0 p^{n-1} l_0 + i_1 p^{n-2} l_1 + \cdots + i_{n-1} l_{n-1}}{p^n} \right\rceil.$$

Here u_i and l_i is $(i+1)$-th upper ramification jump *and* lower ramification jump *of L/K respectively.*

Singularities

By using the formula above, we can compute the right hand side of (1). Since the stringy motives contain information on singularities of the quotient varieties, we can study singularities in terms of the $\boldsymbol{v}$-function. As a corollary, we get the following simple criterion for an indecomposable $\mathbb{Z}/p^2\mathbb{Z}$-representation:

Theorem 3 (T.–Yasuda [3, Theorem 5.16]) *Assume that $G = \mathbb{Z}/p^2\mathbb{Z}$ and V is an indecomposable G-representation of dimension d $(p+1 < d \le p^2)$. Then*

$$X = V/G \text{ is } \begin{cases} \text{terminal,} \\ \text{canonical,} \\ \text{log canonical,} \\ \text{not log canonical} \end{cases} \text{if and only if } \begin{cases} d \ge 2p+1, \\ d \ge 2p, \\ d \ge 2p-1, \\ d < 2p-1. \end{cases}$$

Note that the indecomposable $\mathbb{Z}/p^2\mathbb{Z}$-representation V of dimension d is not effective if and only if $d \le p$, has pseudo-reflections if and only if $d = p+1$.

References

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